Is the principle of contradiction a consequence of $x^2 = x$?

Jean-Yves Beziau

Abstract. According to Boole it is possible to deduce the principle of contradiction from what he calls the fundamental law of thought and expresses as $x^2 = x$. We examine in which framework this makes sense and up to which point it depends on notation. This leads us to make various comments on the history and philosophy of modern logic.

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Keywords. Boole, principle of contradiction, law of thought, symbolic logic, subtraction, universal logic, Boolean algebra, square of opposition

Many people think that it is impossible to make algebra about anything except number. This is a complete mistake … The use of algebra is to free people from bondage.

Mary Everest Boole[1]

Philosophy and Fun of Algebra (1909)

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References

[1] Mary Everest Boole (1832-1916) was the wife of George Boole (1815-1864) and the niece of George Everest (1790-1866) whose name was given to the Mount Everest. She had many interests, in particular for mathematical pedagogy. She contributed to the preparation of *The laws of thought* and also to the early death of her husband.

1. Symbolic variations on a proof

In the *Laws of Thought* (Boole 1854), George Boole presents a proposition, and a proof of it, according to which the principle of contradiction is a consequence of what he calls "the fundamental law of thought" and that he expresses as $x^2 = x$. Such a result is generally neither presented nor discussed in present books or courses of logic. However this is a very interesting result both from a philosophical and mathematical point of views, a result due to a key figure of the history of logic. [2]

In this paper we examine the meaning and validity of this proof, presenting 7 versions of it, ranging from the original one to the reconstruction of it in classical propositional logic as it is nowadays conceived through versions in first-order logic, Boolean algebra, algebra of sets. This is a way in particular to study the interplay between notations and conceptual frameworks.

Paul Halmos famously claimed: *the best notation is no notation* (Halmos 1970). The full quotation runs as follows: "The best notation is no notation; whenever it is possible to avoid the use of a complicated alphabetic apparatus, avoid it. A good attitude to the preparation of written mathematical exposition is to pretend that it is spoken. Pretend that you are explaining the subject to a friend on a long walk in the woods, with no paper available; fall back on symbolism only when it is really necessary."

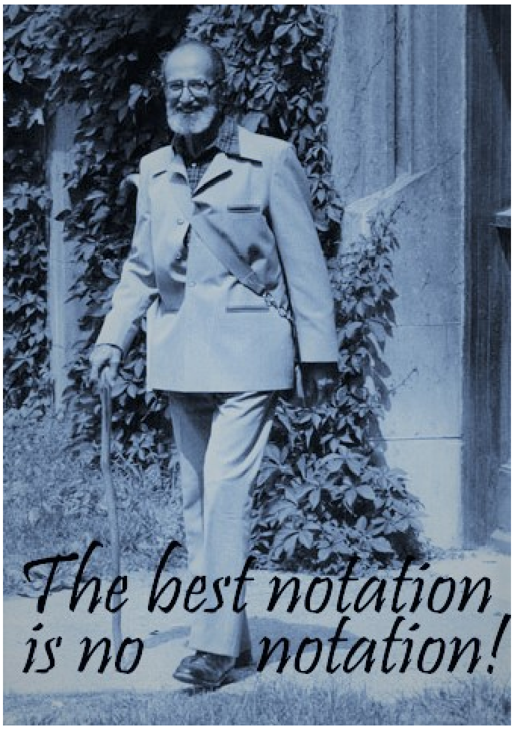

FIGURE 1. Paul Halmos: walking notation

However, modern logic, which was symbolically named "symbolic logic",[3] strongly depends on notation. Beyond different ways of writings there are different ways of thinking. But there is an even more fundamental phenomenon: modern logic is not just a new way of

[2] Our paper is not (only) for Boolean specialists and does not require special acquaintance with his work. People wanting to know more about Boole can consult: Corcoran 2006, Bar-Am 2008, Jacquette 2008.

[3] There is the famous book by John Venn with this title (Venn 1881). But we don't know if he himself was the first to coin the expression. The expression *Association for Symbolic Logic* and *Journal of Symbolic Logic* were not coined by direct reference to the work of John Venn, the idea is simply that this kind of logic is using signs which are not purely alphabetical (cf. Ducasse and Curry 1962).

writing, it is a new way to use notation, a new semiology, in continuity with the semiotic changes introduced by the British school of *symbolic algebra* from which Boole emerged. Model theory developed by Tarski in the 1950s can be considered as the final step of a semiotic evolution started by Peacock with symbolic algebra at the beginning of the 19th century. This evolution can be described in different ways (cf. Durand-Richard, 2007, 2009). Here we give a more precise and more specific account of this evolution from an internal point of view focusing on one proof, showing that the signs of this proof can, not only be interpreted in different ways, but can work differently.

We start (section 2) by a discussion about the relation between mathematical formalism and thought. This is another aspect of the variation discussed here to which we will come back in the final part of our paper (section 8), after presenting the classical propositional version of the proof, discussing the various formalizations/formulations of the principle of contradiction, a key principle since the beginning of the history of logic with Aristotle, that can be understood in many different ways through symbolization. And this principle has been challenged at the time of modern logic in particular with the development of paraconsistent logic, an important family of non-classical logics,

2. A strange proposition

PROPOSITION IV of Chapter III of the *Laws of Thought* of Boole is formulated as follows:

> … the principle of contradiction … is a consequence of the fundamental law of thought, whose expression is $x^2 = x$

This is strange for two main reasons. Firstly, before Boole nobody considered that $x^2 = x$ was a fundamental law of thought. Secondly, it is not clear how we can derive the principle of contradiction from this fundamental law. In this paper we will discuss only this second aspect. But this is related to the first point, because if we can derive a principle as fundamental as the principle of contradiction from $x^2 = x$, this is a good reason to consider that $x^2 = x$ is a fundamental law.[4]

This proposition can be seen as establishing connections between two heterogeneous fields, on the one hand metaphysics, on the other hand mathematics. We can draw a parallel with the famous controversy in St Petersburg in the 1770s (cf. Fig.2), when Leonard Euler (1707-1783) told Denis Diderot (1713-1784):

Sir! $(a + b^n)/n = x$, therefore God exists, respond!

Since algebra sounded for Diderot like Hebrew, he was ridiculed and ran back to Paris with the tails between his legs.[5]

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[4] Note that Boole did not use quotation marks, he did not write : the fundamental law of thought, whose expression is "$x^2 = x$". What we are talking about is the proposition / thought corresponding to "$x^2 = x$", which can also alternatively be denoted by "$xx = x$" or "$x.x = x$" or "$x \times x = x$" depending on the sign we are using for multiplication.

[5] This story was reported by different people, so there are slightly different versions, see (Gillings 1954). Funny enough the name "Denis Diderot" was given to the *University Paris 7* known for having one of the best departments of mathematics in France (including the strongest group of logicians in France and also of categoricists; this is where the author of this paper did a Master (on paraconsistent logic) and a PhD (on universal logic). The reason to choose Diderot's name was because the University of Paris 7 is the most pluridisciplinary university of Paris, this fits well with the fact that Diderot was an Encyclopedist. After May 1968, the University of Paris was split into many universities, each having a specialty and a number. Paris 7 was created in 1971 and the name "Denis Diderot" officially given to it in 1994. In 2019 it will merge with the *University of Paris 5* (Paris Descartes University). Diderot + Descartes: a winning strategy!

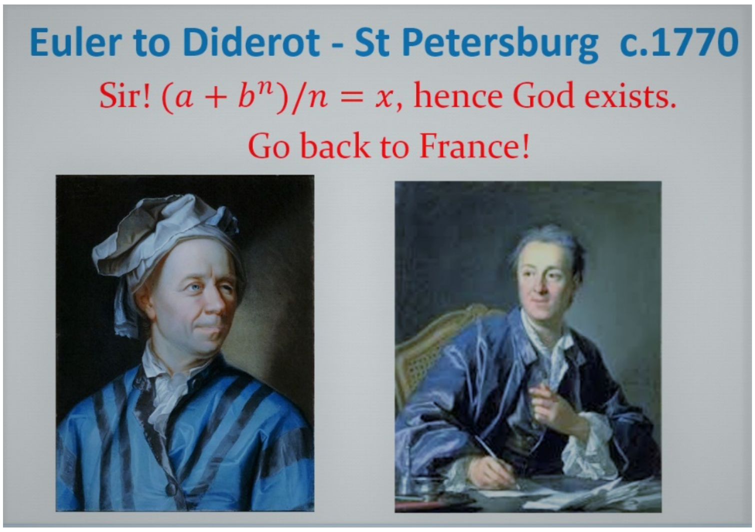

FIGURE 2. Euler, Diderot and God

A remake of this story (cf. Fig.3) could be as follows: during a meeting in the 1850s in London, George Boole told Karl Marx (1818-1883):

Sir! $x^2 = x$, therefore there are no contradictions, respond!

Since algebra sounded for Marx like Arabic, he was ridiculed and ran back to Germany with the tails between his legs.[6]

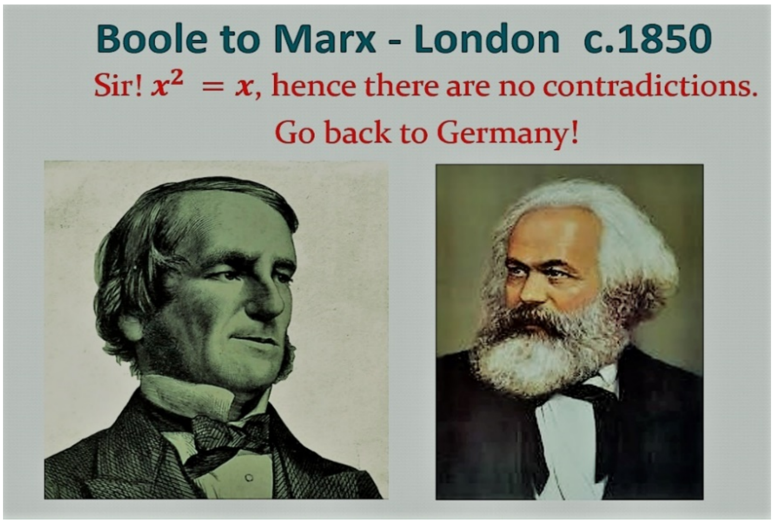

FIGURE 3. Boole, Marx and Contradiction

[6] As it is known Marx moved to London in 1849. Boole was living at this time in Ireland. But we may imagine a meeting organized by Queen Victoria in London with both of them and other great intellectuals such as Stanley Jevons, Lewis Carroll, Friedrich Engels (who joined Marx in England in 1849). Marx and Engels were quite weak in mathematics although the latter tried to use mathematics to support their politico-philosophical theories. The bad quality of these writings was one of the reasons why Jean van Heijenoort (1912-1986), nicknamed "JvH") abandoned Marxism after having been 10 years the secretary and bodyguard of Trotsky in Mexico (cf. Feferman 1993, 2012) –see his 1948 essay on Engels and mathematics (Heijenoort 1948). When he was young, JvH had stopped his studies of mathematics in Paris , to join Trotsky. He was chosen because of his knowledge of many languages, knowledge he later used to work in the history of modern logic. JvH has promoted the myth of Frege as the founder of modern logic, cf. his book *From Frege to Gödel* (Heijenoort 1967) and his paper "Historical development of modern logic" (Heijenoort, 2012). He had no interest for Boole. According to his PhD student I.Anellis, this is due in particular to a personal conflict he had at Brandeis University with one of his colleagues (see Anellis 1994). The late Irving Anellis edited a special issue of *Logica Universalis* in 2012 for the centenary of JvH.

Someone may dismiss the Euler-Diderot encounter as an anecdote. But is it really anecdotal? And what is an anecdote? "A short amusing or interesting story about a real incident or person" according to *Oxford Dictionary*.[7] Let us emphasize that on the one hand something can be at the same time funny and important, these are not incompatible features; and that on the other hand this incident was not specially funny for Diderot. But "Anecdotal" has a negative meaning: qualifying an event as anecdotal means that it is secondary. This secondary aspect can be related to shortness. But it is not because an event is short that it is secondary. Can we consider that César saying *Alea iacta est* and crossing the Rubicon is anecdotal? Maybe if we take this event isolated. But from the perspective of the full story, it is far to be an anecdote. Similarly the Euler/Diderot encounter can be considered just as an anecdote or as a meaningful event part of an important story, the history of intellectual life, of human culture and science. Our paper is a way to have a look at this encounter as much more than an anecdote.[8]

Someone may wonder how we can seriously do that relating this real encounter to a fictional meeting between Boole and Marx one century later. Our idea is to use the power of analogy, analogy in the original Greek sense of analogical proportion: *A* is to *B* as *C* is to *D*.[9] Nowadays nobody, even a good Christian, would consider that Euler's equation is a serious proof of the existence of God.[10] On the other hand, can a Marxist or a non-Marxist says that Boole's proof is not a serious proof of the validity of the principle of contradiction, entailing the non-existence of contradictions?[11]

Marx and his followers, in particular Engels and Mao, considered that contradictions were everywhere, that contradiction was the motor of nature and society (cf. the famous Mao's 1937 pamphlet entitled *On Contradiction*). For this reason formal logic was at first rejected in Soviet Union. It was considered part of bourgeois thinking, because it is based on the principle of non-contradiction.[12]

In the 20th century, systems of logic in which the principle of contradiction is not valid were developed. They are called "paraconsistent logics" (see Batens et al. 2000, Carnielli et al. 2002, Beziau et al. 2007, 2015, Tanaka et al. 2013).

There had been no direct relations with Marxism. Newton da Costa, considered as the main developer of paraconsistent logic, was mainly motivated by paradoxes of set theory.[13]

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[7] https://en.oxforddictionaries.com/definition/anecdote

[8] "Anecdote" means in Greek *unpublished*. The word was used as a name for the unpublished book written in the 6th century by Procopius of Caesarea about the emperor Justinian, depicting not so glorious aspects and behaviors of the emperor and his wife. Hence the ambiguous meaning of the word: at some point this book was considered as a series of gossips contrasting with the official story written by Procopius himself. But nowadays this book is considered as a serious complementary story that Procopius was not allowed to tell, the "official" story he wrote, *The Wars of Justinian*, being itself a distortion of reality.

[9] A formal theory of analogical proposition has recently been developed by H.Prade and G.Richard (2013), for a more general approach to analogy see (Beziau 2018).

[10] On the other hand formal proofs of the existence of God, in particular using logical systems, have been presented, in particular by Gödel ; see the special issue of *Logica Universalis* on Logic and Religion edited by J.-Y.Beziau and R.Silvestre (2017).

[11] It is usual to use the expression "principle of contradiction" as a shortening for the expression "principle of non-contradiction", this is indeed what Boole did. But people favoring contradiction prefer to use the full length version.

[12] see Bocheński, Küng, Lobkowicz, 1961.

[13] The Russian logician Nikolai Vasiliev (1880-1940) and the Polish logician Stanislaw Jaśkowski (1906-1965) are considered as forerunners of paraconsistent logic, but none of them was a defender/promoter of dialectical materialism, In fact their work became popular only at the end of communism - see (da Costa et al. 1995).

But in paraconsistent logic Boole's proof is indeed not valid as we shall see in the final part of our paper.

Let us point out that if on the one hand, Boole, a major figure in the development Modern Logic, did reinforce the principle of non-contradiction, with the proposition under examination here, by deriving it form a more elementary and obvious principle, on the other hand Jan Łukasiewicz, also a major figure in the history of Modern Logic, started his work by a critical analysis of the arguments presented by Aristotle to defend the principle of non-contradiction (Łukasiewicz 1910).

In the perspective of the analogy between Euler/Diderot and Boole/Marx, one may think that Boole's proposition/proof is a kind of trick, transforming symbolic signs into thoughts arriving at a surprising result, Boole being like a magician throwing dices into a hat becoming then a dove flying out of the hat.[14]

To show that it is not a trick is not necessarily obvious. This is one of the aims of this paper. In fact if we put the above proof of PROPOSITION IV of Chapter III of the *Laws of Thought* as an exercise for a student, or even a professor, of the University of Foxbridge she will probably not be able to present such a proof. She may even claim that there is no such a proof because it is false.

The examination of the proof of PROPOSITION IV is interesting for many reasons. It is related to the question of notation. Does this proof depend on the concept of notation? Does any proof depend on notation? Of course everybody knows that it is not the same to compute using Roman numerals or Indo-Arabic numerals, but on the one hand computing is not the same as proving (even if we consider that proofs are recursive), and on the other hand the result, independently of the notation, is the same.

And what is the relation between notation and conceptual framework? Two notations may differ only due to superficial aspects. We can change the fonts of our text, choosing bigger fonts or another type of fonts. We can also transcribe a Russian text using Latin alphabet; this does not mean we are translating Russian into Latin. And a pictogramatic language does not work in the same way as an alphabetic language – however we can still claim that a Chinese basically thinks in the same way as a Russian.

But sometimes, change of notation goes hand to hand with change of conceptual framework. And when we go to mathematical notation, it is much more than a change of language; it is a change of way of thinking.[15]

Using mathematics to deal with logic, Boole changed the theory of reasoning. He also changed the way of reasoning[16] and changed mathematics, introducing non numerical algebra, going out of the sphere of quantities. In his 1898 book, *A Treatise on Universal Algebra*, Whitehead wrote the following (p.35): "The Algebra of Symbolic Logic is the only known member of the non-numerical genus of Universal Algebra", with the footnote: "This algebra in all essential particulars was invented by Boole, cf. his work entitled, *An Investigation of the Laws of* Thought, London, 1854."

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[14] The connection between magic and logic appears through the work of Lewis Carroll, a "symbolic" logician, and also through the work and talents of the late Raymond Smullyan (Smullyan 2015).

[15] (Serfati, 2005) is a very interesting book about the development of mathematical symbolism.

[16] About the difference between logic as reasoning and logic as the theory of reasoning, see our paper "Logic is not logic" (Beziau 2010).

Boole did not start out of nothing. His work was developed under the influence of the British school of symbolic algebra promoted by George Peacock (1791-1858) and Duncan Farquharson Gregory (1812-1844) – about this school, see in particular the works of Marie-José Durand-Richard, and also (Grattan Guinness 2000) and (Peckhaus 2009).[17]

In the next section we will have a close look at Boole's proof in its original context.

3. Analysis of the original three-step derivation of Boole

The center of our attention is the following proof in three steps:

(1B)	$x^2 = x$
(2B)	$x - x^2 = 0$
(3B)	$x(1-x) = 0$

TABLE 1 – BOOLE ORIGINAL DERIVATION

Each line of the proof is as in the original work of Boole. The three equations on the right column are written exactly in this way in the first edition of *The Laws of Thought* on page 49 (III.15).[18] Boole does not present this in a table and does not use the word "proof" to qualify what we call here a "3-step derivation". We are using a table to make the reading easier and to compare with other derivations that we will present and discuss.

The word "derivation" means here *a development of a theorem*. This is item 5a of the definition of "derivation" in *Dictionary.com*[19]*.* It is commonly used in this sense both by specialists of proof-theory and by non-specialists, lay mathematicians. It is more neutral than "proof" and leaves open the degree of analysis and formalism. This is why we will use this word and, as it is also commonly done, we can use it both to speak of the whole process, or to speak of a specific part of it: going from step 1 to step 2 is a derivation.

Boole successively presents these three equations in the order given in TABLE 1. (1B) is presented at the end of PROPOSITION IV and he does not repeat it in a separate line. (2B) is presented at a middle of a line ending with a comma:

$$x - x^2 = 0,$$

and so is (3B) but with a semicolon and "(1)" at the very end on the right side:

$$x(1-x) = 0; \qquad (1)$$

To write these three equations successively in separate lines to express the fact that we go from the first to the third through the second and that all are true is a procedure which is still common in contemporary books of algebra as well as the details of the fonts: italic for the variable, no italics for the numbers, the parentheses, the exponent and the identity sign. It is also common to write something like "(1)" at the beginning or the end of the line, using

[17] We will not discuss here the relation between Boole and Leibniz, in particular the distinction between *Lingua Universalis and Calculus Ratiocinator* (Couturat 1904 is still one of the best presentations of this distinction in Leibniz), nor will we discuss the relation between Boole and computation (see e.g. Friend 2010).

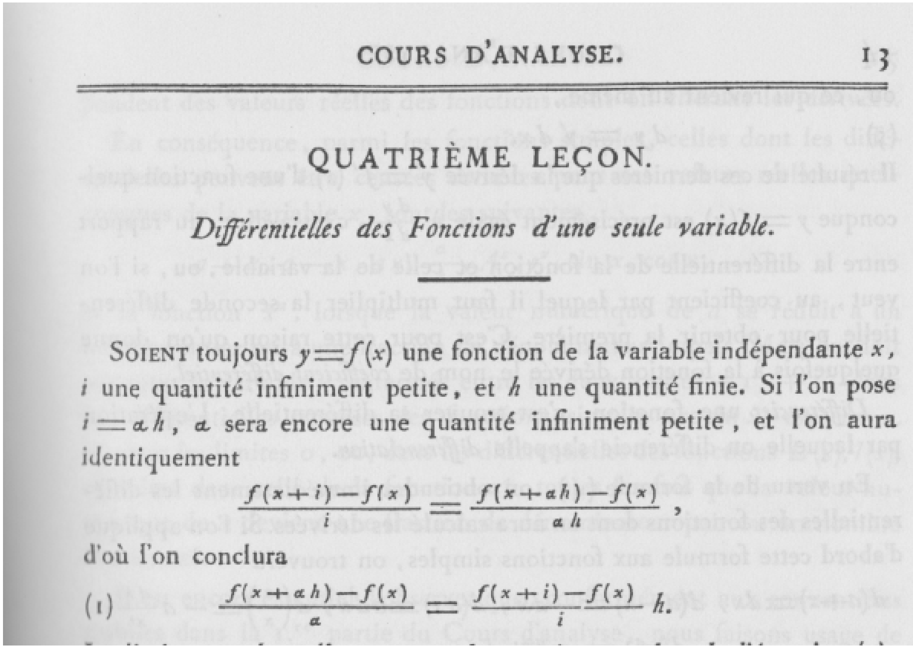
[18] The original version of Boole's *Laws of thought* has been reproduced in the Dover edition with the same pagination. A scan of the original book is available on the internet by *Internet Archive*, a non-profit digital library founded by Brewster Khale. Note also that Boole uses Roman numbers for Chapter and to each paragraph is attributed an Indo-Arabic number. So it is quite easy to make precise references to the Bible of modern logic.

[19] http://www.dictionary.com/browse/derivation?s=t
Dictionary.com is one of the main on-line dictionaries for English language. It was founded in 1995 and is based on the Random House Unabridged Dictionary, with other content from the Collins English Dictionary, American Heritage Dictionary and others.

it as a label for the asserted equation. Centering the equation is also common as well as the use of punctuation marks. The whole thing is like a sentence, the idea to separate the sentences by starting new lines is to emphasize the assertion of each equation, their truth.

Frege introduced the sign "⊢" to make the distinction between a thought and its assertion (cf Frege 1879; for an analysis of it, see Smith, 2009). Before him this distinction was operated by the "lining-device". Frege's sign what adopted by Whitehead and Russell in *Principia Mathematica* (1910-1913) but not by Hilbert who didn't like it and kept using the traditional lining-device (cf Hilbert-Ackerman 1928).

This way of writing (lining-device, italic/non-italic fonts, label) derivations of equations was developed before Boole during the 18th century. At the beginning of the 19th century it was a standard way to write mathematics. See on Fig.4 an extract of Cauchy's famous *Cours sur le Calcul infinitésimal* (1823).


COURS D'ANALYSE. 13

QUATRIÈME LEÇON.

Différentielles des Fonctions d'une seule variable.

SOIENT toujours $y=f(x)$ une fonction de la variable indépendante x, i une quantité infiniment petite, et h une quantité finie. Si l'on pose $i=\alpha h$, α sera encore une quantité infiniment petite, et l'on aura identiquement

$$\frac{f(x+i)-f(x)}{i}=\frac{f(x+\alpha h)-f(x)}{\alpha h},$$

d'où l'on conclura

(1) $$\frac{f(x+\alpha h)-f(x)}{\alpha}=\frac{f(x+i)-f(x)}{i}h.$$

FIGURE 4. Cauchy's *Cours sur le Calcul infinitésimal*

This way of writing has been preserved up to now independently of the development of mathematical logic (showing the weak, not say null, impact on logic upon mathematics, that for many people turns things more complicated without any further advantages).

In a standard contemporary book of algebra it is not necessarily explained what justifies the passage to the second equation from the first and to the third from the second. Students generally learn this as kind of informal rules.

Boole goes from step 1 to step 2, saying "let us write this equation in the form" and then presenting (2B). But after going form step 2 to step 3 on the basis of a "whence", he qualifies both derivations by using the word "transformation", saying "these transformations being justified by the axiomatic laws of combination and transposition (II.13)".

We will not discuss here up to which point Boole was the first to be so precise. But all his work shows a careful analysis of the way mathematicians are proceeding. It would be interesting also to investigate the relation between the procedure described by Boole and the development of the theory of "Rewriting" (about this theory, see e.g. N.Dershowitz and D.Plaisted, 2001).

4. Boolean algebra

If we present this 3-step derivation to a sample student, let say Natasha, she will think that it is part of algebra, not part of logic; in particular not part of propositional logic where there are no equations, no identity sign, except if we consider idiosyncratic constructions by someone like Roman Suszko (see e.g. Bloom and Suszko, 1972).

Which algebra? At first sight Natasha may think that it is about an algebra with 1 and 0 and other numbers on which x is ranging: natural numbers, rational numbers or reals numbers. But looking more closely Natasha will change her mind because $x^2 = x$ is false if we consider that this free way to use the variable x corresponds to universal quantification, which is the standard interpretation (in the sense both of its usual meaning and of the correlated formalized version of it in first-order logic).

What is also standard for a lay mathematician is to consider that the line-device corresponds to truth. So Natasha will only think of this derivation in the context where $x^2 = x$ is true. She will not consider the validity of the derivation independently of the truth of $x^2 = x$, only logicians do that or postmodern mathematicians.

If we consider that x is ranging over natural numbers, $x^2 = x$ is generally false. Just consider that the value of x is Natasha's age (for the sake of privacy we will not reveal it here). This equation is true among natural numbers exactly for two of them: 0 and 1. That is why Natasha will claim: "This is (a derivation in) Boolean algebra".

If her teacher is a bit tricky she will correct her, adding: "Boolean algebra on {0,1} !". We now know that the Boolean algebra on {0,1} is the simplest Boolean algebra, but that a Boolean algebra can have a domain of more than two elements and not necessarily numbers: the power set of any set forms a Boolean algebra.

Let us emphasize that Boole's 3-step derivation can be performed/written exactly in the same way nowadays in standard mathematics. But people have totally forgotten the meaning originally given by Boole to it: derivation of the principle of contradiction from the fundamental law of thought. They generally don't interpret $x^2 = x$ as a fundamental law of thought and $x\,(1 - x) = 0$ as the principle of contradiction.

About the relation between what is now called "Boolean algebra" and Boole's original work, see Hailpern 1981 and Burris 2000. The paper of Hailpern is symptomatically called "Boole's algebra isn't Boolean algebra". But considering the following quotation of Boole, it is reasonable to argue that he conceived what we now called the Boolean algebra on {0,1}:

> We have seen (II. 9) that the symbols of Logic are subject to the special law,
>
> $$x^2 = x$$
>
> Now of the symbols of Number there are but two, viz. 0 and 1, which are subject to the same formal law. We know that $0^2 = 0$, and that $1^2 = 1$; and the equation $x^2 = x$, considered as algebraic, has no other roots than 0 and 1. Hence, instead of determining the measure of formal agreement of the symbols of Logic with those of Number generally, it is more immediately suggested to us to compare them with symbols of quantity *admitting only of the values* 0 *and* 1. Let us conceive, then, of an Algebra in which the symbols *x, y, z*, etc. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established. II.15

5. Boolean algebra from the point of view of first-order logic

Boolean algebra can be considered today from the point of view of classical first-order logic. First-order logic can itself be considered from two perspectives: model theory and proof theory. Sometimes this distinction is presented with a linguistic flavor as a distinction between semantics and syntax. This is rather ambiguous.

If we establish a strong link between syntax and computation, then many things that are called semantics can be called syntax. This is in fact what Chang and Keisler do in their famous book on model theory (Chang and Keisler 1973) qualifying truth-tables as syntax. Moreover they define model theory with the following equation:

$$\textit{universal algebra} + \textit{logic} = \textit{model theory}$$

Although we can enjoy their sense of humor, structures do not reduce to algebras.[20] Model theory is a relation between mathematical structures and syntax, syntax not in the sense of proof-theory but in the sense of formulas and formulations. The study of syntax in this sense has been developed in details by logicians. Mathematicians are rather fuzzy about that.

Garrett Birkhoff's HSP theorem (1935), one of the main theorems in the pre-history of model theory (officially developed by Tarski, 1954-55), states that a class of algebras is closed under homomorphism, subalgebra and product iff it can be defined by a set of equations. It establishes a relation between operations on structures and syntactic expressions. We can re-formulate Chang and Keisler's equation as follows:

$$\textit{mathematical structures} \mathscr{H} \textit{syntax expressibility} = \textit{model theory}$$

We did not use the sign of addition but a sign conveying interaction.

The syntax expressibility of first-order logic is a sophisticated methodology that took many years to be developed (Cf. Hodges 1985-86). Here is how we can present Boole's derivation from this point of view:

(1FOL)	$\forall x \quad x \times x = x$
(2FOL)	$\forall x \quad x - (x \times x) = \mathbb{0}$
(3FOL)	$\forall x \quad x \times (\mathbb{1} - x) = \mathbb{0}$

TABLE 2 – FIRST-ORDER-LOGIC

The question is not to eliminate abbreviation for multiplication but to consider that multiplication and subtraction are not part of the syntax. And nor are "0" and "1", the numerals for the numbers zero and one.

"$\times$" can be interpreted in different ways. Its meaning is fixed by some axioms. A sign similar to "x", a standard symbol for multiplication, is used, because it is the *intended-interpretation*, the same for "$\mathbb{1}$" and "$\mathbb{0}$". Without using the intended-interpretation methodology, the formula (3FOL) should be written as follows:

$$\forall x \quad x \, \mathbb{R} \, (\mathbb{a} \, \mathbb{S} \, x) = \mathbb{b}$$

$\mathbb{R}$ and $\mathbb{S}$ are signs for binary functions subject to various interpretations, $\mathbb{a}$ and $\mathbb{b}$ being signs for objects subject also to various interpretations.

[20] In some sense, algebras are prototypes of mathematical structures. At some point the word "structure" was used as a name for lattice (cf Glivenko 1938). As it is known Bourbaki put the notion of structures at the heart of mathematics. They considered three basic mother structures: algebraic, topological and order structures (see Bourbaki 1948). About the development of abstract algebra and mathematical structures, see Corry 2004.

The question of identity is also tricky; we will not discuss it here (see e.g. Beziau 1996). We could have put "=" and not "=", then identity would have been considered as a primitive notion, not subject to a variety of interpretations.

All that we have said here is much related with model-theory of first-order logic, "interpretations" refers to models. But this can also be viewed from the perspective of proof-theory where "interpretation" is interpreted as the meaning of the operations and objects given by the rules of a proof-theoretical system.

Now that we have put the correct syntax, can we say that the three lines of TABLE 2 form a correct derivation? We can consider this derivation from first-order proof theory or from first-order model theory. In both cases the derivation is not true from a pure logical point of view (meaning of the quantifier and identity), we need to put some further axioms governing the non-logical entities.

Let us consider the derivation from a model-theoretical point of view. According to the notion of semantical consequence promoted by Tarski (1936)[21], (2FOL) follows from (1FOL) iff all models of (1FOL) are models of (2FOL) and (3FOL) follows from (2FOL) iff all models of (2FOL) are models of (3FOL), and going directly to the first line to the third line: (3FOL) follows from (1FOL) iff all models of (1FOL) are models of (3FOL).

This is no true for all models, but it is true for a certain class of models: rings. Ring is a sufficient condition, but not a necessary one. And ring itself is not the correct terminology because a ring is generally defined with multiplication and addition, not with subtraction. A structure can be defined in various equivalent ways. Stone (1935) has proven that an idempotent ring is equivalent to a complemented distributive lattice. An *idempotent ring* is a ring with as additional axiom the formula (1FOL), it also called a *Boolean ring*.

In model theory two structures are said to be equivalent iff they have a common expansion by definition up to isomorphism. The notion of expansion is a notion mixing syntax and semantics. An open question is: how can we conceive a structure independently of a given language, of a given signature?

Instead of interpreting Boole's three-step derivation from the point of view of a class of models. We can consider it within a particular model, i.e. within the Boolean algebra on {0,1}:

(1BOL)	$b \times b = b$
(2BOL)	$b - (b \times b) = 0$
(3BOL)	$b \times (1 - b) = 0$

TABLE 3 – BOOLEAN ALGEBRA ON {0,1}

We have changed the name of the variable, renaming "x", "b", to emphasize that its domain of variation is {0,1}. We also have used back usual signs for multiplication, subtraction, identity, zero and one, because here the derivation is performed in the context of concrete operations and objects. In this case each line of the derivation can be justified as true by considering the two possible instantiations of the variable.

Let us note that in mathematics it is common to use the same sign, for example "0" to denote different objects, e.g. zero as a natural number, zero as an integer. This can be justified by the fact that one is an element of a substructure of the other structure. This phenomenon is indeed explicitly explained by model theory which justifies this "abus de langage".

[21] J.Corcoran and J.M. Miguel Sagüillob (2011) have argued that this notion was not really presented by Tarski in his 1936's paper but only during his Berkeley period (Tarski 1954-55).

6. Algebra of sets

We can say that the most famous *model* of Boolean algebra is the Boolean algebra on {0,1}. Other famous *models* are algebras of sets. It is possible to prove that the power set of any set forms a Boolean algebra and there is the famous Stone representation theorem (1936) stating that every Boolean algebra is isomorphic to a field of sets.

Let us rewrite the formulas of TABLE 2 with this intended interpretation. We have then the following table:

(1FAS)	$\forall x$	$x \cap x = x$
(2FAS)	$\forall x$	$x \setminus (x \cap x) = \varnothing$
(3FAS)	$\forall x$	$x \cap (\mathbb{U} \setminus x) = \varnothing$

TABLE 4 – FIRST-ORDER ALGEBRA OF SETS

The intended interpretation of $\cap$, $\setminus$, $\varnothing$, $\mathbb{U}$ are respectively intersection, subtraction, the empty set and the universal set, i.e. the set of all objects. We can rewrite TABLE 4 as follows:

(1AS)	$A \cap A = A$
(2AS)	$A \setminus A \cap A = \varnothing$
(3AS)	$A \cap (\boldsymbol{U} \setminus A) = \varnothing$
(4AS)	$A \cap \bar{A} = \varnothing$

TABLE 5 – INFORMAL ALGEBRA OF SETS

The derivation from (1AS) to (3AS) is generally performed using intuitive properties of set-theoretical operations, like when doing derivations with equations on numbers. This kind of derivation can be interpreted as "working in the model".

We have added a fourth line introducing the standard notion of set complement, that can be considered as an abbreviation or a *primitive* notion.

$A \cap \bar{A} = \varnothing$ seems a reasonable formulation of the principle of contradiction. This can be supported by the confusion between contrariety and contradiction (see Beziau 2016). But if we are using the word contradiction more precisely as defined by the theory of the square of opposition, the following axiom should be added $A \cup \bar{A} = \boldsymbol{U}$. For example, a square cannot be a circle, the intersection between the sets of squares and circles is empty, but something can be neither a circle nor a square, that's why these two notions are not contradictory. A right example is: straight line and curve; they are contradictory concepts because obeying the additional axiom (see Beziau 2015). If we call the principle of contradiction not the fourth line, but the third line, this avoids the confusion. Considering that $\bar{A} = \boldsymbol{U} \setminus A$, we indeed have $A \cup \bar{A} = \boldsymbol{U}$.

Boole had clearly in view this algebra of sets, but he was using the word "class" rather than "set"[22]. He wrote:

[22] Marx also used at this time this word; "Klasse" in German. In set theory the word used in German is not "Klasse" but "Menge". The main difference between an algebra of sets and set theory is that an algebra of sets can be seen/defined only as/with relations between sets, without talking about the elements of this set. As pointed out by Badiou, the idea of class developed by Marx works also like that, the individuals don't matter (see Badiou, 1988, Part II, , more precisely: p.105 in the English edition).

If x represent any class of objects, then will $1-x$ represent the contrary or supplementary class of objects, i.e. the class including all objects which are not comprehended in the class x. For greater distinctness of conception let x represent the class men, and let us express, according to the last Proposition, the Universe by 1; now if from the conception of the Universe, as consisting of "men" and "not-men," we exclude the conception of "men," the resulting conception is that of the contrary class, "not-men." Hence the class "not-men" will be represented by $1-x$. And, in general, whatever class of objects is represented by the symbol x, the contrary class will be expressed by $1-x$. III.14

Boole identifies (3AS) with (4AS), and (4AS) with the principle of contradiction, as stated by Aristotle (which is in fact, in view of the theory of oppositions, originated by the Stagirite himself, is a confusion between contradiction and incompatility):

the equation (1) $x\,(1-x)\;=0$ thus expresses the principle, that a class whose members are at the same time men and not men does not exist. In other words, that it is impossible for the same individual to be at the same time a man and not a man. Now let the meaning of the symbol x be extended from the representing of "men," to that of any class of beings characterized by the possession of any quality whatever; and the equation (1) will then express that it is impossible for a being to possess a quality and not to possess that quality at the same time. But this is identically that "principle of contradiction" which Aristotle has described as the fundamental axiom of all philosophy. "It is impossible that the same quality should both belong and not belong to the same thing.. . . This is the most certain of all principles. Wherefore they who demonstrate refer to this as an ultimate opinion. For it is by nature the source of all the other axioms." (Metaphysics III; 3). III.15

Couturat in his famous booklet *L'algèbre de la logique* published in France in 1905 and translated few years later (1914) in English presented a system with two interpretations, conceptual and propositional interpretations, and made the following comments:

the algebra in question, like logic, is susceptible of two distinct interpretations, the parallelism between them being almost perfect, according as the letters represent concepts or propositions. Doubtless we can, with Boole and Schröder, reduce the two interpretations to one, by considering the concepts on the one hand and the propositions on the other as corresponding to assemblages or classes; since a concept determines the class of objects to which it is applied (and which in logic is called its extension), and a proposition determines the class of the instances or moments of time in which it is true (and which by analogy can also be called its extension). Accordingly the calculus of concepts and the calculus of propositions become reduced to but one, the calculus of classes, or, as Leibniz called it, the theory of the whole and part, of that which contains and that which is contained. But as a matter of fact, the calculus of concepts and the calculus of propositions present certain differences, as we shall see, which prevent their complete identification from the formal point of view and consequently their reduction to a single calculus of classes. Accordingly we have in reality three distinct calculi, or, in the part common to all, three different interpretations of the same calculus. In any case the reader must not forget that the logical value and the deductive sequence of the formulas does not in the least depend upon the interpretations which may be given them, and, in order to make this necessary abstraction easier, we shall take care to place the symbols C. I. (conceptual interpretation) and P. I. (propositional interpretation) before all interpretative phrases. These interpretations shall serve only to render the formulas intelligible, to give them clearness and to make their meaning at once obvious, but never to justify them. They may be omitted without destroying the logical rigidity of the system. (Couturat 1914, p.2)

In the next section, we will see how the propositional interpretation can be seen today from the point of view of the theory of models.

7. Classical propositional logic

Let us finally examine the relation between Boole's original derivation and classical propositional logic (hereafter CPL). Not CPL as conceived by Boole, but as it is nowadays understood.

It is common to claim that CPL is a Boolean algebra. What does it mean exactly? Model theory permits a clear formulation of the problem: can CPL be considered as a first-order structure which is a model of the axioms of a Boolean ring or an equivalent structure?

Having in mind CPL as intended-interpretation we can rewrite TABLE 2 as follows:

(1P)	$\forall x$	$x \wedge x \dashv\vdash x$
(2P)	$\forall x$	$x - (x \wedge x) \dashv\vdash \bot$
(3P)	$\forall x$	$x \wedge (\mathbb{T} - x) \dashv\vdash \bot$

TABLE 6 – FIRST-ORDER CLASSICAL PROPOSITIONAL LOGIC

This is a very simple way to proceed. We have done nothing else that changing the signs. But it is not usual to consider CPL as a model of FOL. Someone may think this is strange or even says: you cannot do that! But there are no vicious circles here and this method will be justified and will fully make sense in our next step (TABLE 7).

This can be seen as a way to formalize the meta-logic of CPL within FOL. Dana Scott (cf. Scott 1974, pp.412-413) suggested that one motivation of Tarski to develop model theory was to carry out the study of non-classical logics, in particular many-valued logics, within classical logic. And in fact this can be done considering non-classical propositional connectives as functions within (classical) FOL, having different kinds of interpretations.

The mix between logic and meta-logic is typical of the Polish school and in a letter to Otto Neurath posthumously published in 1992, Tarski explained that this was promoted in particular by Łukasiewicz at the beginning of the 1920s in Warsaw and that he explained this perspective to Gödel during a travel to Vienna. This was certainly a decisive step towards Gödel's arithmetization of syntax, pivotal for the famous theorems to which his name is attached.

As we have simplified TABLE 4 into TABLE 5, we can simplify TABLE 6 into TABLE 7:

(1P)	$p \wedge p \dashv\vdash p$
(2P)	$p - (p \wedge p) \dashv\vdash \bot$
(3P)	$p \wedge (\top - p) \dashv\vdash \bot$
(4P)	$p \wedge \neg p \dashv\vdash \bot$

TABLE 7 – CLASSICAL PROPOSITIONAL LOGIC

Similarly as in TABLE 5 we have introduced a 4th line. In TABLE 5, we used "A" as a sign for set, in table 5 "p" as a sign for proposition, similarly as "n" is generally used as a sign for natural number. In TABLE 2, TABLE 4 and TABLE 6 we have used "x" as a sign for variable, more precisely a *bound variable*. In TABLE 1 "x" is used as a free variable, there are no quantifiers.

As it is known quantifiers were introduced explicitly only after Boole (By Peirce, Schröder, Frege, the symbol "$\forall$" itself having been introduced by Gentzen). The letter "x" as a free variable was introduced by Descartes. It is common in CPL to talk about "propositional variable", but it is not necessarily clear what is meant by this expression. Sometimes "p" is used as a sign for atomic proposition only. Here we are using it as a sign for any proposition, atomic, or molecular, in other words, for any formula.

An important difference between TABLE 7 and TABLES 3 and 5 is that $\dashv\vdash$ is not identity. The symbol " $\dashv\vdash$ " is not so much used in logical books. "$p \wedge p \dashv\vdash p$" can be seen as an abbreviation of "$p \wedge p \vdash p$ and $p \vdash p \wedge p$". The relation denoted by " $\dashv\vdash$ " is called *logical equivalence*. In case of CPL it is possible to show that it is a congruence relation and then we can consider the factor structure which is called a *Tarski-Lindenbaum* algebra or sometimes just a *Lindenbaum* algebra. This construction can be generalized to any logical structure where logical equivalence is a congruence relation. These logics are called *self-extensional.* In the factor structure $\dashv\vdash$ is then identity. What can be said is that, in case of CPL, the Tarski-Lindenbaum is a Boolean algebra.[23]

Now can we say that TABLE 7 is about classical propositional logic? There are different ways to present CPL, not necessarily equivalent (see Beziau 2001). In the Polish tradition it is common to consider a propositional logic as a structural consequence relation. This is what we are doing here, using the sign "$\vdash$"to denote the consequence relation, i.e. a relation between theories (sets of formulas) and formulas.

In TABLE 7 we have $\top$ and $\bot$. They are not always considered in CPL but their use is known. The origin of the sign "$\top$" is probably due to the fact that "$\top$" is the first letter of "True". However the constant "$\top$" does not denote truth but logical truth, it is something which is always true: $\vdash \top$. The sign "$\bot$" is not the first letter of "False", it was probably chosen to graphically express the contrast between truth and falsity. However "$\bot$" does not denote falsity but logical falsity. It is something which is always false, this can be expressed by: $\bot \vdash p$. $\top$ and $\bot$ can also be used in other logics than CPL, they are not characteristic features of CPL. At some point Tarski considered that the presence of $\bot$ was an axiom for any logic (see Beziau 2006). It would be completely confusing here to use "1" and "0" instead of "$\top$" and "$\bot$", since nowadays "1" and "0" are used to denote truth-values which are not part of the syntax of CPL.

What about "$-$"? It is rarely used in CPL. It is called "logic subtraction" and has been studied in particular by Haskell Curry (1952).[24] Let us consider the following four connectives:

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[23] This construction was performed for CPL by Tarski (1935), but not exactly in this way, he was considering the connective of bi-implication, not the relation of logical equivalence. Nevertheless, we can say that Tarski proved that CPL is a Boolean algebra. The first step is to consider that connectives from a syntactic point of view are functions and that that the set of formulas is an algebra, an absolutely free algebra (idea attributed to Lindenbaum).

[24] The connective of subtraction has been revived in recent years through dual intuitionistic logic (see e..g. Shramko 2005) and bi-intuitonistic logic (see Bellin et al. 2014).

p	q	$p \longrightarrow q$	$\neg(p \longrightarrow q)$	$p \longleftarrow q$	$\neg(p \longleftarrow q)$
0	0	1	0	1	0
0	1	1	0	0	1
1	0	0	1	1	0
1	1	1	0	1	0

The relations between them can pictorially be described by Blanché’s hexagon of opposition (see Blanché 1957, Béziau 2012, Wybraniec-Skardowska 2016), cf. Fig. 5.

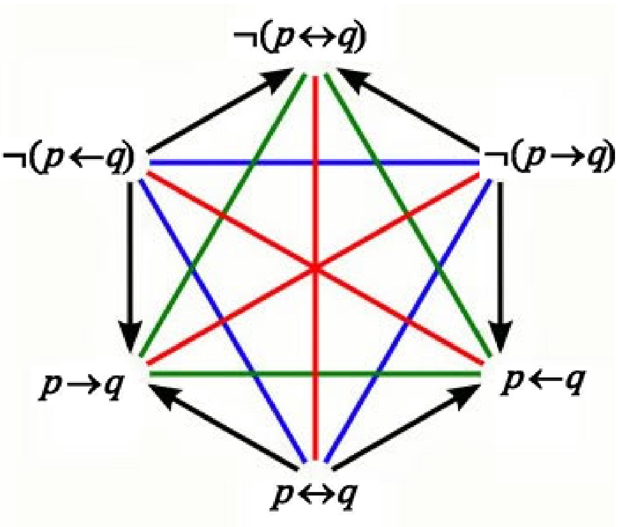

FIGURE 5. Blanché’s hexagon for implications

The following table shows that if we want to define $\neg p$ as $\top - p$ we have to consider that subtraction corresponds to $\neg(p \longleftarrow q)$:

p	$\top$	$p \longrightarrow \top$	$\neg(p \longrightarrow \top)$	$p \longleftarrow \top$	$\neg(p \longleftarrow \top)$	$\neg p$
0	1	1	0	0	1	1
1	1	1	0	1	0	0

We can alternatively use the sign “$\nleftarrow$” for subtraction and “ $\nrightarrow$ ”, as a sign for the negation of implication. We know that the following 18 sets of two connectives are truth-functionally complete:

$\vee$ $\neg$	$\rightarrow$ $\nleftrightarrow$	$\leftrightarrow$ $\nrightarrow$
$\wedge$ $\neg$	$\leftarrow$ $\nleftrightarrow$	$\leftrightarrow$ $\nleftarrow$
$\rightarrow$ $\neg$	$\rightarrow$ $\bot$	$\rightarrow$ $\nrightarrow$
$\leftarrow$ $\neg$	$\leftarrow$ $\bot$	$\leftarrow$ $\nrightarrow$
$\nrightarrow$ $\neg$	$\nrightarrow$ $\top$	$\rightarrow$ $\nleftarrow$
$\nleftarrow$ $\neg$	$\nleftarrow$ $\top$	$\leftarrow$ $\nleftarrow$

If we therefore interpret the connectives of TABLE 7 classically we can say that this is a derivation in CPL.

An interesting question to investigate is the following: which is the weakest subsystem of classical logic in which it is possible to perform this derivation?

This is related to two other points. In CPL, we can go from (1p) and (3P) and vice-versa. If we follow Boole's idiosyncrasy, it means that in CPL we can go from the fundamental law of thought (FLT) to the principle of contradiction (PNC) and back. From the point of view of CPL therefore FLT is not more fundamental than PNC. But maybe we can construct a system according to which PNC is a consequence of FLT but not conversely, or the other way round. The second point is to investigate up to which point (3P) or/and (4P) is the correct formalization of PNC and, in case of other formulations of PNC, what are their relations with FLT.

8. Formalizations of the principle of non-contradiction

Can we consider that

$$p \wedge \neg p \dashv\vdash \bot$$

is a good formulation of the principle of non-contradiction?

From the right to the left it is a consequence of the fundamental axiom for ⊥, known in Latin as *ex falso sequitur qudo libet*, that can be written in symbols as follows:

$$\bot \vdash q$$

So let us examine the other direction:

$$p \wedge \neg p \vdash \bot$$

Considering the *ex falso* and transitivityy we have:

$$p \wedge \neg p \vdash q$$

This sometimes is called e*x contradictione sequitur quod libet*. But is $p \wedge \neg p$ really a contradiction?

First, let us note that $p, \neg p \vdash q$ is not equivalent to $\vdash \neg(p \wedge \neg p)$. The two are independent as it can be shown by the following table:

p	$\neg p$	$p \wedge \neg p$	$\neg(p \wedge \neg p)$
0	1	0	1
½	½	½	½
1	0	0	1

This is a table using a three-valued logical matrix. If ½ is considered as designated, $\vdash \neg(p \wedge \neg p)$ is valid but not $p, \neg p \vdash q$. If ½ is considered as non-designated, $p, \neg p \vdash q$ is valid but not $\vdash \neg(p \wedge \neg p)$.

The standard definition of paraconsistent logic is based on the rejection of $p, \neg p \vdash q$ (see e.g. Beziau 2000) but there are paraconsistent logics in which also $\vdash \neg(p \wedge \neg p)$ is not valid. We have recently called such logics g*enuine paraconsistent* logics and developed two three-valued genuine paraconsistent logics (see. Beziau 2016 and Hernández-Tello et al. 2017).

In classical logic, we have the validity of both formulations of the principle of contradiction. But can we consider that the two together form the principle of contradiction? Not in the sense of contradiction as in the square of opposition, because they are independent of the principle of excluded middle, this means they don't exclude contrariety. The situation is parallel to the one at the hand of our section 5. A formulation of the principle of contradiction that is compatible with the square definition is:

$$p \wedge (\top - p) \vdash \bot$$

and, as claimed by Boole, this is derivable from:

$$p \wedge p \dashv\vdash p.$$

9. The gap, the bridge and the snark

Up to now very little has been said by commentators about PROPOSITION IV of Chapter III of the *Laws of Thought*. Badesa (2004) wrote the following:

> Boole attributes special importance to the law $x^2 = x$ (that is, $xx = x$) because from it the principle of non contradiction $x\,(1 - x) = 0$ is deduced, but above all because he considers it to be characteristic of the operations of the mind, as it is the only one of the basic laws that does not hold in the algebra of numbers. (Badesa, 2004, p6)

Badesa discusses neither the validity of Boole's derivation, nor the validity of the formulation of the principle of non-contradiction as $x\,(1 - x) = 0$. His presentation is a mix of summary and vulgarization of Boole's ideas from a contemporary perspective. This approach in fact is very common among historians of logic and is certainly useful. Other authors have tried to establish the connection between what is nowadays understood as Boolean algebra and Boole's original work, showing the differences and the resemblances.

Here we have done something else: we have shown how Boole's derivation of the principle of contradiction (PNC) from the fundamental law of thought (FLT) can be presented within classical propositional logic (TABLE 7). This is an original contribution, which is not necessarily obvious because for doing that one has to use a non-standard formulation of CPL. We are speaking here of a non-standard formulation of CPL because it is based on a framework using logical equivalence and considering as primitive the binary connective of subtraction and the zero-ary connectives *verum* and *falsum*. These connectives are known back to Peirce and Post, but CPL has barely been studied from this perspective. An extentionalist may say it is all the same, on the other hand a true intensionalist will defend the idea that different point of views on the same thing are certainly interesting. There are non-denumerable formulations of the axiom of choice and many of them are interesting by themselves. Knowing the axiom of choice is not just knowing one formulation of it.

The link between the derivation of PNC from FLT within CPL and Boole's original presentation can be understood through Boolean algebra formulated in first-order logic, CPL being one model of such a theory. Our presentation/discussion develops the relation between FOL and CPL, a topic that has not yet been much examined. Our paper is a way to stimulate further research in that direction.

And also in a more philosophical direction: the connection and interaction between philosophical ideas and formalizations. Boole's PROPOSITION IV and the corresponding proof may appear just as tricks. To analyze the situation we have to clearly distinguish two things:

(1) The interpretations given by Boole both to the fundamental law of thought and the principle of contradiction.
(2) The proof itself.

In the present paper, we have focused on the second point, and we have shown that the proof is valid from the point of view of CPL. Strictly speaking, it does not necessarily mean that the original derivation presented by Boole is valid or true in itself. The *gap* between the two formulations is big and it is certainly not just a question of notation. Notation can in fact give the illusory impression that these two formulations of the proof are very similar. Notation can serve as a bridge leading from one to the other, but we have to be conscious that this *bridge* connects two quite different landscapes.

Our version of Boole's derivation in CPL can be qualified as a "reconstruction". This kind of reconstruction is interesting because it gives some new inspirations for further developments. This is in the line of Łukasiewicz's approach to history of logic which gave a great impulsion for the advancement of logic in the 20th century, in particular it was pivotal for the development of many-valued logic. And also in the spirit of Łukasiewicz's original work we have dealt with the principle of contradiction, its understanding and its formalizations. We intend to further develop this line of research by examining in which systems exactly it is possible to derive PNC from FLT, viewed as $p \wedge p \dashv\vdash p$, and conversely. This is in the spirit of universal logic, the idea being to see the exact framework, presuppositions and hypotheses behind a theorem.

Another very interesting question, we have here only marginally dealt with, connected to the second part of the point (1) above, is the relation between $x^2 = x$ and the so-called fundamental law of thought. The statement of Badesa is highly ambiguous: this law is different from the laws of numbers, but this does not explain why it is characteristic of our thought, any law different from the law of numbers would be characteristic of our thought in this sense.[25]

We intend to develop this point in further investigations in particular making the connection between $p \wedge p \dashv\vdash p$ and the famous verse of *The Hunting of the Snark* by Lewis Carroll:

What I tell you three times is true

Acknowledgements

I first tackled this question, among many others, during my talk "Semiotics of modern logic" presented at the workshop *History and Philosophy of Logic Notations* organized at the Tallinn University of Technology, Tallinn, Estonia, August, 1-2, 2015. I am grateful to the organizer of this workshop, Amirouche Z. Moktefi, for the invitation.

I later on further developed this question and the main content of the present paper was presented as an invited talk at the 12TH INTERNATIONAL CONFERENCE - LOGIC TODAY: DEVELOPMENTS AND PERSPECTIVES, June 22–24, 2016, St. Petersburg, Russia. This explains why the famous story about Euler and Diderot went to my mind. Reference to Marxism in this context is also interesting because of the October revolution which took place in 1917 in this city, later on re-baptized "Leningrad". I would like to thank Elena Lisanyuk and Ivan Mikirtumov from the Department of Logic of the University of St Petersburg for the invitation. My travel to Russia was partly supported by the Santander University Exchange Program between Russia and Latin America.

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[25] Moreover this law is not different from the law of numbers if we consider only 0 and 1, the key to Boolean algebra on {0,1}.

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